Topological entropy of autonomous flows

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The topological entropy of autonomous flows is evaluated by a method based on symbolic dynamics and on the thermodynamic formalism for nonlinear dynamics. This technique, which applies to all generalized entropies K_q, reproduces a well-known formula for the metric entropy and clarifies the relationship between a flow and the associated Poincaré maps, beyond the straightforward case of periodically forced nonautonomous systems. Numerical results for the Lorenz and Rössler systems are presented and verified with an independent estimator. [S1063-651X(96)50211-4]

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One of the primary indicators of chaos in a dynamical system is the topological entropy K_0 [1], which measures the exponential increase of the number of orbits belonging to sets with suitable separation properties [2], as a function of the orbit length. Therefore, K_0 quantifies the ''richness'' of solutions of the system. Furthermore, it enjoys a few other properties, such as invariance under smooth coordinate changes, which make its estimation a common step in the analysis of chaotic systems. The calculation is, however, mostly restricted to maps [3] or to periodically forced flows. For this reason, the present paper focuses on the definition and evaluation of the topological entropy K_0 for general flows and experimental signals. The method is based on a thermodynamic approach that applies to the whole generalized entropy function K_q [4], which accounts for the fluctuations of local entropies and includes K_0 as a particular case.

In the analysis of smooth flows with low-dimensional attractors, time t is usually discretized by means of a Poincaré section Σ [5]. If this is properly chosen, the ''original'' dynamics described by the vector field $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},t)$, with $\mathbf{x}\in\mathbb{R}^d$ and $\mathbf{f}$ a nonlinear function, is equivalent to that of the resulting Poincaré map $\mathbf{y}_{n+1}=\mathbf{F}(\mathbf{y}_n,n)$ on the surface of section Σ, where $\mathbf{y}_n\in\Sigma$ and $\mathbf{F}$ is another nonlinear function. The correspondence is indeed straightforward in the case of nonautonomous systems with a periodic time dependence of $\mathbf{f}$. In particular, the generalized entropies K_q of the flow equal those of the map divided by the period T. For autonomous flows, instead, $\mathbf{f}$ is time independent and the orbit intersects Σ at irregularly spaced times t_n. Nevertheless, a relation between flow and map entropies still holds, although only for the metric entropy K_1, whereby the average value of the return time $T_n=t_n-t_{n-1}$ is involved [6].

In this work, a relation valid for generic entropies K_q, $\forall q$, is proposed. The method extends a grand-canonical formulation of the thermodynamic formalism for maps, based on symbolic dynamics. Applications to the Lorenz and Rössler systems are presented for $q=0$, 1, and 2, and verified by an independent estimate obtained from a suitable average of the flows' local expansion rates. The analysis makes use of the sole information that is usually available from experimental time series.

The generalized entropies K_q can be defined in a homogeneous way for maps and flows considering the probability $P(\epsilon,t,\mathbf{x}_0)$ of finding an orbit of length t (either a real or a natural number) within a maximum distance ϵ from the orbit $\{\mathbf{x}_\tau,\ 0\leqslant\tau\leqslant t\}$ originating at the point $\mathbf{x}_0$ (the symbol $\mathbf{x}$ denoting, in the following, phase-space points for both flows and maps). In the chaotic regime, this quantity, which refers to the natural invariant measure m [7] of the system, is usually assumed to scale as

$$P(\epsilon,t;\mathbf{x}_0)\sim\epsilon^{\alpha(\mathbf{x}_0)}e^{-\kappa(\mathbf{x}_0,t)t}, \tag{1}$$

for $t\to\infty$ and $\epsilon\to 0$, where $\alpha(\mathbf{x}_0)$ is the local dimension of the ϵ neighborhood of $\mathbf{x}_0$ and $\kappa(\mathbf{x}_0,t)$ is the local entropy of the given portion of the trajectory [8]. The generalized dimensions and entropies D_q and K_q, respectively, are then defined through the moments of P as

$$\langle P(\epsilon,t;\mathbf{x})^{q-1}\rangle\sim\epsilon^{(q-1)D_q}e^{-t(q-1)K_q}, \tag{2}$$

in the limit $t\to\infty$, $\epsilon\to 0$ [8]. For $q\to 1$, one obtains the information dimension D_1 and the metric entropy K_1 [7] and, for $q=0$, the box dimension D_0 [9] and the topological entropy K_0 [1].

Clearly, if the flow crosses the surface of section at times $t_n=nT$, with a fixed T, $K_q^f=K_q^m/T$, where the superscripts refer to flow and map, respectively. For generic systems (i.e., either autonomous or aperiodically forced nonautonomous), $K_1^f=K_1^m/\langle T\rangle$ [6], while nothing is known for $q\neq 1$. A direct evaluation of K_q from Eq. (2), on the other hand, is rather cumbersome because of the large statistical fluctuations of P for large t and of the slow convergence of the finite-t estimates of K_q for $\epsilon\ll 1$, especially when phase transitions occur in the thermodynamic representation of nonlinear dynamics [10,11] and in experimental systems [12].

An alternative, direct definition of the topological entropy K_0^f for flows is also available [13]:

$$K_0^f=\lim_{T\to\infty}\frac{1}{T}\ln N(T), \tag{3}$$

where $N(T)$ is the number of (unstable) periodic orbits of length at most T. Also this relation, however, is hardly usable in practice because only low-order orbits can usually be located by analyzing a time series (especially if experimental). Better results can be obtained by ordering the dynamics hierarchically and applying predictive algorithms that accelerate the convergence of the estimates to the asymptotic limit.

To this aim, one introduces a partition of the map's phase space consisting of a finite number b of disjoint subsets B_k $(k=0,\ldots,b-1)$ so that an orbit $\omega=\{\mathbf{x}_0,\ldots,\mathbf{x}_n\}$ is associated with the symbol sequence $S=s_0s_1\cdots s_n$, where $s_i\in\{0,1,\ldots,b-1\}$ is the label of the element visited at time t_i. One further requires the partition to be generating, i.e., such that infinite symbol sequences correspond to individual points in phase space [14]. Since the map $\mathbf{F}$ preserves the natural measure m, one has $m(B_k)=m[\mathbf{F}^{-1}(B_k)]$, $\forall\ k$, and the probability $P(S)$ of sequence S equals $m[B_{s_0}\cap\mathbf{F}^{-1}(B_{s_1})\cap\cdots\cap\mathbf{F}^{-n}(B_{s_n})]$. Accordingly, the generalized entropy is defined through the canonical partition sum

$$Z_n(q)=\sum_S P^q(S)\sim e^{-n(q-1)K_q}, \tag{4}$$

which runs over all sequences S of length n. Because of the constancy of n, this formula cannot be immediately extended to flows by letting n take on real values. One first needs a grand-canonical formulation in which variable-length sequences appear in the sum.

Relation (4) descends from a tree representation of the symbolic dynamics in which the b symbols label the offsprings of the root, pairs of symbols the nodes of the second level, and concatenations of n symbols those of the generic nth level. Since the dynamics usually folds phase space incompletely over itself, some transitions between partition elements are forbidden and the corresponding sequences are pruned off the tree. In such cases, or whenever the probabilities $P(S)$ exhibit large variability, it is convenient to parse the symbolic signal generated by the map into variable-length words. This technique, which is called "coding," is currently implemented by information compression algorithms, like Lempel and Ziv's [15], to reduce the redundancy of the signal. For example, if $S=00$ is forbidden in the binary case $b=2$, the dynamics yields concatenations of the two words $w_1=1$ and $w_2=01$: in terms of them, a complete binary tree is recovered. The number and the lengths of such words may vary greatly from case to case, depending on the system and on the coding method.

The tree formed after the coding presents concatenations W of l basic words w_i at the lth level. Whatever symbolic regrouping has been chosen, it is necessary to reformulate the thermodynamic sum (4) in order to account for the variability of the word lengths. Let us, therefore, introduce the grand-partition function

$$\Omega_l(z;q)\equiv\sum_W P^q(W)z^{\ell(W)}, \tag{5}$$

where W is a level-l word and $\ell(W)\geqslant l$ its length (number of symbols). The term $z^{\ell(W)}$ provides a detailed compensation for the generally exponential decrease of $P(W)\sim\exp[-\kappa\ell(W)]$ with $\ell(W)$. In fact, the series converges if $z<z(q)$, for $l\to\infty$, where $z(q)$ is the convergence radius. In particular, by setting $z=z(q)=\exp[(q-1)K_q]$ into Eq. (5), equating the result to 1, and letting $q\to 1$, one obtains the metric entropy

$$K_1=\lim_{l\to\infty}-\sum_W P(W)\ln P(W)\Bigg/\sum_W P(W)\ell(W) \tag{6}$$

for a variable-length coding [16]. In general, rather than determining $z(q)$ by imposing the arbitrary constraint $\Omega_l(z;q)=1$ (which would introduce prefactor errors), one compares the partition functions at two consecutive levels:

$$\sum_W P^q(W)z^{\ell(W)}=[\lambda(z;q)]^{-1}\sum_{W'} P^q(W')z^{\ell(W')}, \tag{7}$$

where $W=w_0w_1\ldots w_l$ is a generic level-l word and $W'=Ww_{l+1}$ any of its offsprings. In the limit $l\to\infty$, $z(q)$ is determined from the relation $\lambda[z(q);q]=1$. A standard procedure [17,16] finally permits one to rewrite Eq. (7) as an eigenvalue equation for a generalized transfer matrix $\mathbf{T}_l(z,q)$, the entries of which read, at level $l+1$,

$$T_{w_0w_1'\cdots w_l';w_1\cdots w_{l+1}}\equiv\sigma^q(Ww_{l+1})z^{\ell(Ww_{l+1})-\ell(W)}\delta_{w_1'w_1}\cdots\delta_{w_l'w_l}, \tag{8}$$

where $\sigma(Ww_{l+1})\equiv P(Ww_{l+1})/P(W)$ is a term of the level-l probability scaling function [18], which consists of the symbolically ordered l-word conditional probabilities. Consideration of these rates is tantamount to performing an order-l word-to-word Markov approximation of the system's scaling dynamics, with considerable improvement of the convergence of the estimates. With this formulation, the generalized entropy is given by

$$K_q=\frac{\ln z(q)}{q-1}, \tag{9}$$

where $z(q)$ is the value of z for which the largest eignevalue $\lambda_1(z;q)$ of $\mathbf{T}$ equals 1, in the limit $l\to\infty$; i.e., it is the solution of

$$\det[\mathbf{T}_l(z,q)-\mathbf{I}_l]=0 \tag{10}$$

for $l\to\infty$, where the index l again indicates the hierarchical level. The polynomials obtained in this way coincide with those of the transition matrices for subshifts of finite type (i.e., for finite-order Markov processes), up to a possible overall factor z^p, $p\in\mathbb{N}$.

Evidently, this approach can be readily extended by letting $\ell(W)$ take on real values and, in particular, the actual temporal extension (briefly called "length," in the following) of the orbits of the flow associated with the map's symbolic sequence W. In this way, Eq. (9) yields either K_q^m or K_q^f depending on whether $\ell(W)$ equals the number $|W|$ of symbols in W or the length of a corresponding orbit of the flow, respectively. Since each word W identifies a finite region B_W of the map's phase space, which shrinks to a point only when $|W|\to\infty$, the value of $\ell(W)$ may be chosen with some freedom: for example, either as the average or the shortest return time of the flow trajectory to B_W, or as the length of the periodic orbit with symbol sequence W (or a cyclic permutation of it), if this exists (some word W, in fact, may not be periodically extendible, so that B_W contains no periodic point belonging to an order-$|W|$ orbit). In any of these cases, the overall map (flow) may be interpreted as the union of local maps, each defined on a B_W and advancing the orbit by a time $\ell(W)$. Such a construction is called ''induction'' [6]. Within this scheme, Eq. (6) reproduces Abramov's theorem [6]: in fact, the metric entropy of the map (flow) equals that of the induced map (obtained via the word partitioning) divided by the average orbit length at level l, in the limit $l\to\infty$. Clearly, if $\ell(W)=T|W|$, for all W, one solves Eq. (10) for z^T and recovers the trivial case $K_q^f=K_q^m/T$: if, in addition, no prohibitions occur at level l, the polynomial (10) admits the same zeros as at the previous level. Variations of T with the sequence W in the flow may either accelerate or slow down the convergence to the limit with respect to the map: appearance of short (long) T's increases (decreases) K_q.

The method has been applied in the cases $q=0$, 1, and 2 to the Lorenz system [19]

$$\dot{x}=-10(x-y),$$
$$\dot{y}=-y+28x-xz, \tag{11}$$
$$\dot{z}=-\beta z+xy,$$

at $\beta=8/3$ and $\beta=1$ and to the Rössler system [20]

$$\dot{x}=-y-z,$$
$$\dot{y}=x+0.2y, \tag{12}$$
$$\dot{z}=0.2-5.7z+xz.$$

In the former, the Poincaré section has been chosen as $\Sigma=\{(x,z): x=y,\ \ddot{x}\ \mathrm{sgn}(x)<0\}$ (see [12] for further details); in the second, as the plane $x=0$ with $\dot{x}>0$. Since the method is proposed for numerical as well as for experimental data, the generating partition has been approximated by first extracting all unstable periodic orbits of order up to 9 from an embedded scalar time series and by requiring that different symbolic labels be attributed to all periodic points on the section [12]. This yields a binary (ternary) partition for the Lorenz system at $\beta=8/3$ ($\beta=1$), defined, respectively, by $x=0$ and $x=\pm0.2$ in the (x,z) plane, and a binary partition for the Rössler system, defined by $y=-6.74$ in the (y,z) plane [21]. Among the three possibilities proposed above for the evaluation of the lengths $\ell(W)$, I have chosen the one based on the unstable periodic orbits, since these have been already detected prior to the construction of the partitions and are quite easily identifiable even from experimental time series. To test the robustness of the method, the lengths $\ell(W)$ assigned to symbol sequences without periodic extensions have been approximated by means of differences such as $\ell(0010)\approx\ell(001\,011)-\ell(11)$ or $\ell(0221)\approx\ell(010\,221\,110)-\ell(01)-\ell(110)$. This procedure has been originally proposed in [22] for other observables.

For the Lorenz system at $\beta=8/3$, the finite-size estimates $K_0^f(l)$ of K_0^f, obtained by solving Eq. (10) at resolution level l, yield $K_0^f(3)\approx0.914$, $K_0^f(4)\approx0.916$, $K_0^f(5)\approx0.910$, and the asymptotic value $K_0^f=0.910\pm0.005$. It must be noted that the map's topological entropy $K_0^m\approx\ln 2$ divided by the average return time $\langle T\rangle\approx0.747$ would give the larger value 0.928, which indeed coincides with $K_0^f(2)$. For comparison, the first average Lyapunov exponent is $\langle\lambda_1\rangle=0.9056\pm0.0001$, which agrees with the value of K_1^f given by relation (6), and $K_2^f=0.901\pm0.005$.

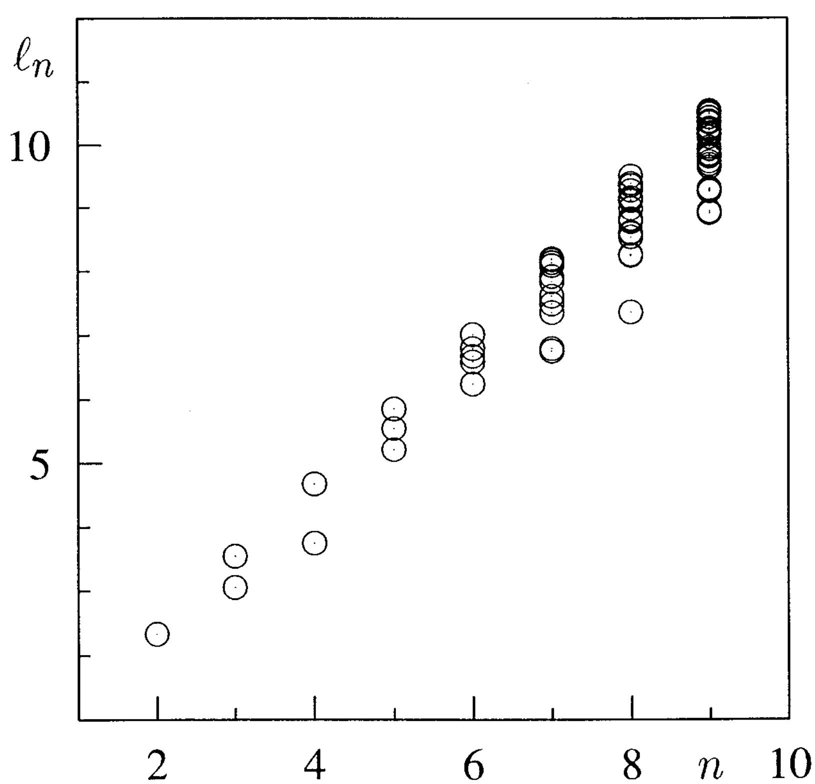

FIG. 1. Plot of the temporal extensions ℓ_n of the unstable periodic orbits of the Lorenz system at $\beta=1$ as a function of the order n. Notice how the spread in their values increases with n.

At $\beta=1$, the analysis has been carried out using both the original three symbols 0, 1, and 2, and a code ϕ consisting of the four primitive words (0, 2, 10, 12): the latter, indeed, ''incorporates'' the prohibition of the word 11 [12] and yields a more compact tree. The occurrence of further prohibitions at all levels $2<l\leqslant 9$ slows down the convergence considerably with respect to the previous case for both the map and the flow. In fact, using code ϕ, $K_0^f(2)\approx 0.89$ and $K_0^f(3)\approx 0.83$ are still far from the best estimate of $K_0^f=0.640\pm 0.005$. The broad range in which the lengths of equal-order periodic orbits of the flow are found also hinders the analysis. The lengths l_n are plotted as a function of the order (number of symbols in the associated symbol sequence) n in Fig. 1. The equality between K_1^f and $\langle\lambda_1\rangle\approx 0.5$ is again confirmed, as well as Abramov's formula. Notwithstanding the larger disuniformity of this attractor (difference between K_0 and K_1), the map's topological entropy, $K_0^m=0.713\pm 0.002$, divided by $\langle T\rangle\approx 1.11$ is closer to K_0^f than for $\beta=8/3$. Finally, $K_2^f=0.36\pm 0.01$.

For the Rössler system, the method yields $K_0^f(2)\approx 0.093$, $K_0^f(3)\approx 0.089$, and the asymptotic prediction $K_0^f=0.0890\pm 0.0005$, while the ratio between $K_0^m=0.500\pm 0.015$ and $\langle T\rangle\approx 5.86$ is smaller than K_0^f. The attractor is also quite nonuniform, since $K_1^f=\langle\lambda_1\rangle\approx 0.071\,05$ differs substantially from K_0^f. The level-three approximation $K_1^f(3)=0.0765$ is less accurate than the corresponding one at $q=0$. Even slower convergence has been observed at $q=2$, where $K_2^f(3)=0.0672$ is still far from the best estimate $K_2^f=0.58\pm 0.01$.

All values of K_0^f have been compared with the estimate [10]

$$\tilde{K}_0^f=\lim_{t\to\infty}[\ln\langle\mu_1(t)\rangle]/t, \tag{13}$$

where the local multiplier $\mu_1(t)=\exp(\lambda_1 t)$ is the expansion factor of nearby points along the unstable manifold of the flow over a time t. Values of t up to 180 have been considered, and very good agreement was found with the grand-canonical approach. In the Lorenz system at $\beta=1$, the convergence law $[\ln\langle\mu_1(t)\rangle]/t\sim K_0^f+a\exp(-\gamma t)$ has been observed, with $\gamma\approx 2.3\times 10^{-2}$.

In this work, I have presented a grand-canonical approach to the evaluation of dynamical entropies K_q^f of generic flows, which is not only a practical technique to obtain more precise values than with phase-space averages or with a standard definition valid for $q=0$, but also a theoretical tool to extend the definition of K_q^f to all q's. Further applications to numerical and experimental data from an NMR laser with delayed feedback are planned. A systematic study of the convergence properties of the method (for both maps and flows) will be presented elsewhere.

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